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Normal Coordinates in Kähler Manifolds and the Background Field Method

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Abstract

Riemann normal coordinates (RNC) are unsuitable for Kähler manifolds since they are not holomorphic. Instead, Kähler normal coordinates (KNC) can be defined as holomorphic coordinates. We prove that KNC transform as a holomorphic tangent vector under holomorphic coordinate transformations, and therefore they are natural extensions of RNC to the case of Kähler manifolds. The KNC expansion provides the manifestly covariant background field method preserving the complex structure in supersymmetric nonlinear sigma models.

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1 Introduction

The equivalence principle ensures that general coordinate transformations on curved space-time do not change any physics, so one can consider appropriate coordinates which render an application the simplest. Riemann normal coordinates (RNC) are one of such coordinates in Riemann manifolds [1, 2, 3]. They are defined as coordinates along geodesic lines starting from a chosen point. Hence any point in a patch of RNC has one-to-one correspondence to a tangent vector at the chosen point.

In most of superstring theory, extra dimensions of the higher-dimensional space-time are compactified to a Calabi-Yau manifold [4], which is a Ricci-flat Kähler manifold. This can be described by conformally invariant supersymmetric nonlinear sigma models in two dimensions, whose target spaces are Kähler manifolds [5]. For perturbative (or non-perturbative) analyses we need to expand the Lagrangian in terms of fluctuating fields around the background fields [6]. To do this, what is the most suitable for these analyses is a generally covariant expansion which preserves the complex structure of the target space. RNC provide a generally covariant expansion, but they are *not holomorphic*, whereas Kähler normal coordinates (KNC) give us such an expansion [7]. KNC are defined as coordinates satisfying some gauge conditions on the derivatives of the metric without recourse to geodesics [8].

In this paper, we prove that KNC transform as a holomorphic tangent vector, therefore they are a natural extension of RNC to the case of Kähler manifolds. The KNC expansion of the Lagrangian is a manifestly covariant expansion under holomorphic coordinate transformations of the target space. The relation between RNC and KNC is also shown: they are distinct by terms proportional to the curvature tensor and its covariant derivatives, so they coincide only in flat space. We also give the KNC expansion of tensor fields which can be applied to the KNC expansion of the Lagrangian with a potential term or a higher derivative term.

This paper is organized as follows. In section 2, after a short review of the RNC expansion in Riemann manifolds, we show that RNC in Kähler manifolds are not holomorphic and therefore are inappropriate for an expansion preserving the holomorphy. In section 3, after recalling the definition of KNC, we discuss basic properties of KNC. We then give the theorem elucidating the geometrical interpretation of KNC. The KNC expansion of tensor fields is also provided. In section 4, we apply KNC to the background field method in supersymmetric nonlinear sigma models on Kähler manifolds. Some applications of the KNC expansion (the Wilsonian renormalization group, low energy theorems of Nambu-Goldstone bosons in four dimensions, and the effective field theory on a domain wall solution) are discussed in section 5. In Appendix A, we summarize the geometry of Kähler manifolds. The relation of RNC and KNC is discussed in detail in Appendix B. We give a proof of the theorem in Appendix C.

2 Riemann Normal Coordinates

First, to compare with Kähler manifolds, we recall some properties of RNC in Riemann manifolds following Ref. [1]. Then we discuss the RNC in Kähler manifolds. It is observed that RNC are not holomorphic coordinates in Kähler manifolds, therefore are not suitable in the cases of Kähler manifolds.

2.1 Riemann Normal Coordinates in Riemann Manifolds

Let $\{x^A\}$ be the coordinates in a Riemann manifold M ($A = 1, \cdots, \dim M$). To define RNC, we choose an expansion point φ^A and consider a geodesic $\lambda^A(t)$ starting from this point, with t being an affine parameter ($0 \le t \le 1$). We consider the end point $\lambda^A(1)$ as a general point $\varphi^A + \pi^A$ in the manifold. The geodesic equation in a Riemann manifold can be written as

$$\ddot{\lambda}^A(t) + \Gamma^A{}_{BC}(\lambda)\dot{\lambda}^B(t)\dot{\lambda}^C(t) = 0\,, \tag{2.1}$$

where a dot denotes differentiation with respect to t, and $\Gamma^A{}_{BC}$ is the connection. The geodesic may be expanded in powers of the affine parameter according to

$$\lambda^A(t) = \sum_{N=0}^{\infty} \frac{1}{N!}\lambda^{A(N)}(0)t^N\,, \tag{2.2}$$

where $\lambda^{A(N)}$ is the N-th derivative of the geodesic and with the initial condition

$$\dot{\lambda}^A(0) \equiv \xi^A\,, \tag{2.3}$$

where ξ^A is a tangent vector at the point φ^A. Here, ξ^A is actually tangent to the geodesic. Recursive use of the geodesic equation (2.1) gives relations

$$\lambda^{A(N)}(t) = -\Gamma^A{}_{B_1 B_2 \cdots B_N}(\lambda)\dot{\lambda}^{B_1}(t)\dot{\lambda}^{B_2}(t)\cdots\dot{\lambda}^{B_N}(t)\,. \tag{2.4}$$

Here coefficients $\Gamma^A{}_{B_1 B_2 \cdots B_N}$ are defined by

$$\Gamma^A{}_{B_1 B_2 \cdots B_N} = \nabla_{B_1}\nabla_{B_2}\cdots\nabla_{B_{N-2}}\Gamma^A{}_{B_{N-1}B_N} \tag{2.5}$$

where ∇ denotes the covariant derivative acting on only lower indices of the connection. For instance $\Gamma^A{}_{B_1 B_2 B_3}$ is defined by

$$\Gamma^A{}_{B_1 B_2 B_3} = \nabla_{B_1}\Gamma^A{}_{B_2 B_3} = \partial_{B_1}\Gamma^A{}_{B_2 B_3} - \Gamma^C{}_{B_1 B_2}\Gamma^A{}_{C B_3} - \Gamma^C{}_{B_1 B_3}\Gamma^A{}_{B_2 C}\,. \tag{2.6}$$

We thus obtain the coefficients in (2.2):

$$\lambda^A(t) = \varphi^A + \xi^A t - \sum_{N=2}^{\infty}\frac{1}{N!}\Gamma^A{}_{B_1 B_2 \cdots B_N}|_\varphi \xi^{B_1}\xi^{B_2}\cdots\xi^{B_N}t^N\,, \tag{2.7}$$

where an index φ indicates quantities evaluated at the initial expansion point φ^A. Since the end point of the geodesics is $\varphi^A + \pi^A = \lambda^A(1)$, we have

$$\pi^A = \xi^A - \sum_{N=2}^{\infty}\frac{1}{N!}\Gamma^A{}_{B_1 B_2 \cdots B_N}|_\varphi \xi^{B_1}\xi^{B_2}\cdots\xi^{B_N}\,. \tag{2.8}$$

This can be regarded as a coordinate transformation, and the RNC are defined by inverting this equation: $\xi^A = \xi^A(\pi)$. Therefore there is one-to-one correspondence between a tangent vector in the tangent space at φ and a point in a patch of RNC around φ.

Now let us discuss the properties of RNC. Since any geodesic can be written as $\lambda^A(t) = \xi^A t$ in RNC[1], the expansion of RNC itself by the tangent vector ξ^A gives the relations

$$\bar{\Gamma}^A{}_{(B_1 B_2 \cdots B_N)}|_\varphi = 0\,, \tag{2.9}$$

[1]The degrees of freedom in general coordinate transformations preserving $x^A = \varphi^A$, $\pi^A = x^A - \varphi^A = c\pi'^A + \sum_{N=2}^{\infty} c^A_{B_1\cdots B_N}\pi'^{B_1}\cdots\pi'^{B_N}$, coincide with the number of coefficients $\Gamma^A{}_{B_1\cdots B_N}$ in (2.7) in each order. Hence there exist coordinates such that any geodesics from the origin become straight lines: they are RNC.

where a bar denotes quantities in RNC and parentheses $(\cdots)$ denote the symmetrization of indices: $T_{(A_1A_2\cdots A_N)} = \frac{1}{N!}(T_{A_1A_2\cdots A_N} + T_{A_2A_1\cdots A_N} + \cdots)$. These conditions for RNC are equivalent to

$$\partial_{(B_1}\partial_{B_2}\cdots\partial_{B_{N-2}}\bar{\Gamma}^A{}_{B_{N-1}B_N)}|_\varphi = 0\,. \tag{2.10}$$

In RNC, any tensor can be expanded by a tangent vector ξ^A easily, using identities (2.9) or (2.10). For instance the metric tensor g_{AB} can be expanded as [1]

$$\begin{aligned} g_{AB}(x) = g_{AB}|_\varphi - \frac{1}{3}R_{ACBD}|_\varphi\xi^C\xi^D - \frac{1}{3!}D_E R_{ACBD}|_\varphi\xi^C\xi^D\xi^E \\ -\frac{1}{5!}\left(6D_C D_D R_{AEBF} - \frac{16}{3}R_{CBD}{}^G R_{EAFG}\right)|_\varphi\xi^C\xi^D\xi^E\xi^F + O(\xi^5)\,, \end{aligned} \tag{2.11}$$

in which we have not written any bars; once we obtain an expansion of a tensor in RNC, it can be regarded as an expansion by a tangent vector. Hence it is a tensor equation and holds in any coordinates.

The RNC expansion can be applied to the background field method of nonlinear sigma models [1, 2]. For another derivation of RNC, see Ref. [3].

2.2 Riemann Normal Coordinates in Kähler Manifolds

As in Riemann manifolds, we consider RNC in a Kähler manifold M. Let $\{z^i, z^{*i}\}$ be coordinates in the Kähler manifold ($i = 1, \cdots, \dim_{\mathbf{C}} M$). We consider a geodesic $\lambda^i(t)$ with an affine parameter t ($0 \le t \le 1$), starting from a point $\lambda^i(0) = \varphi^i$ and ending on a point $\lambda^i(1) = \varphi^i + \pi^i$. The geodesic equation in a Kähler manifold is given by

$$\ddot{\lambda}^i(t) + \Gamma^i{}_{jk}(\lambda, \lambda^*)\dot{\lambda}^j(t)\dot{\lambda}^k(t) = 0\,, \tag{2.12}$$

and its complex conjugate, where a dot denotes differentiation with respect to t. In the same way, we can obtain the expansion of $\lambda^i(t)$ in terms of the tangent vectors ξ^i and ξ^{*i}. The first few orders are given by

$$\begin{aligned} \lambda^i(t) = \varphi^i + \xi^i t - \frac{1}{2}\Gamma^i{}_{j_1j_2}|_\varphi\xi^{j_1}\xi^{j_2}t^2 - \frac{1}{3!}\Gamma^i{}_{j_1j_2j_3}|_\varphi\xi^{j_1}\xi^{j_2}\xi^{j_3}t^3 \\ -\frac{1}{3!}R^i{}_{j_1k_1^*j_2}|_\varphi\xi^{j_1}\xi^{j_2}\xi^{*k_1}t^3 + O(t^4)\,, \end{aligned} \tag{2.13}$$

in which ${\Gamma^i}_{j_1j_2j_3}$ is defined by

$$ {\Gamma^i}_{j_1j_2j_3} \equiv \partial_{j_1}{\Gamma^i}_{j_2j_3} - {\Gamma^l}_{j_1j_2}{\Gamma^i}_{lj_3} - {\Gamma^l}_{j_1j_3}{\Gamma^i}_{j_2l} \equiv \nabla_{j_1}{\Gamma^i}_{j_2j_3} \,. \tag{2.14} $$

It is the restriction of (2.6) to holomorphic indices. The expansion (2.13) can be obtained from the expansion (2.7) in Riemann manifolds, by identifying real coordinates with the holomorphic and anti-holomorphic coordinates: $\{x^A\} = \{z^i, z^{*i}\}$.

The end point $\varphi^i + \pi^i = \lambda^i(1)$ of the geodesic can be expressed by

$$ \begin{aligned} \pi^i = \xi^i - \frac{1}{2}{\Gamma^i}_{j_1j_2}|_\varphi \xi^{j_1}\xi^{j_2} - \frac{1}{3!}{\Gamma^i}_{j_1j_2j_3}|_\varphi \xi^{j_1}\xi^{j_2}\xi^{j_3} - \frac{1}{3!}{R^i}_{j_1k_1^*j_2}|_\varphi \xi^{j_1}\xi^{j_2}\xi^{*k_1} \\ +O(\xi^4)\,. \end{aligned} \tag{2.15} $$

The RNC obtained by inverting this equation depend on both π and π^*: $\xi^i = \xi^i(\pi,\pi^*)$. Hence the coordinate transformation from the holomorphic coordinates z^i to RNC ξ^i is *not* holomorphic: *Riemann normal coordinates are not holomorphic*. Such non-holomorphic terms in the transformation (2.15) appear in conjunction with covariant tensors like the curvature tensor ${R^i}_{j_1k_1^*j_2}$. This is very different from the case of Riemann manifolds.

In summary, we comment that

1. The transformation (2.15) can be directly obtained from the transformation (2.8) in Riemann manifolds with identification of coordinates $\{x^A\} = \{z^i, z^{*i}\}$,
2. All non-holomorphic terms in (2.15) appear with coefficients of covariant tensors, so they exist in general unless the Kähler manifold is flat. (See comments in subsection 3.1 and discussions in Appendix B.)

3 Kähler Normal Coordinates

As shown in the last section RNC are inappropriate for Kähler manifolds, since they are not holomorphic. KNC [7] provide normal coordinates that are holomorphic. After recalling the definition of KNC and giving some discussions in the first subsection, we present the theorem which clarifies the geometric properties of KNC in the second subsection. A proof of the theorem is given in Appendix C. The KNC expansion of tensor fields is also given in the last subsection.

3.1 Definition of Kähler Normal Coordinates

Let $K(z, z^*)$ be the Kähler potential: $g_{ij^*}(z, z^*) = K_{,ij^*}(z, z^*)$, where a comma denotes a partial differentiation with respect to coordinates. Decompose coordinates z^i into an expansion point φ^i and a deviation π^i from it: $z^i = \varphi^i + \pi^i$. We define KNC $\{\omega^i, \omega^{*i}\}$, whose origin coincides with the expansion point $z^i = \varphi^i$, as coordinates such that the quantities $K_{,j^* i_1 \cdots i_N} = g_{i_1 j^*, i_2 \cdots i_N}$ for an arbitrary $N \geq 2$ vanish at the origin of ω^i $(z^i = \varphi^i)$ [8]:

$$\hat{K}_{,j^* i_1 \cdots i_N}(\omega, \omega^*)|_0 = \hat{g}_{i_1 j^*, i_2 \cdots i_N}(\omega, \omega^*)|_0 = 0 \,, \tag{3.1}$$

where a hat indicates quantities in KNC, and the index "0" denotes a value evaluated at the origin of KNC, $\omega^i = 0$. These conditions are equivalent to

$$\partial_{i_1} \cdots \partial_{i_{N-2}} \hat{\Gamma}^j{}_{i_{N-1} i_N}(\omega, \omega^*)|_0 = 0 \,, \tag{3.2}$$

which are similar to the conditions (2.10) for RNC in Riemmann manifolds except for symmetrization of indices. The given coordinates z^i (or π^i) can be transformed to such KNC by the following *holomorphic* coordinate transformation [7]:

$$\begin{aligned} \omega^i &= \pi^i + \sum_{N=2}^{\infty} \frac{1}{N!} [g^{ij^*} K_{,j^* i_1 \cdots i_N}(z, z^*)]_\varphi \pi^{i_1} \cdots \pi^{i_N} \\ &= \sum_{N=1}^{\infty} \frac{1}{N!} [g^{ij^*} K_{,j^* i_1 \cdots i_N}(z, z^*)]_\varphi \pi^{i_1} \cdots \pi^{i_N} \,, \end{aligned} \tag{3.3}$$

where the index φ denotes evaluation at the expansion point, $z^i = \varphi^i$, of the original coordinates.

In KNC, the Kähler potential can be expanded due to (3.1), like

$$\begin{aligned} \hat{K}(\omega, \omega^*) = \hat{K}|_0 + \hat{F}(\omega) + \hat{F}^*(\omega^*) + \hat{g}_{ij^*}|_0 \omega^i \omega^{*j} \\ + \sum_{M,N \geq 2} \frac{1}{M! N!} \hat{K}_{, i_1 \cdots i_M j_1^* \cdots j_N^*}|_0 \omega^{i_1} \cdots \omega^{i_M} \omega^{*j_1} \cdots \omega^{*j_N} \,, \end{aligned} \tag{3.4}$$

where $\hat{F}(\omega)$ is a holomorphic function of ω, so it can be eliminated by a Kähler transformation. It has been shown that all coefficients of the expansion (3.4) are covariant tensors [7]. For instance, the forth order coefficient is $\hat{K}_{,i_1 i_2 j_1^* j_2^*}|_0 = \hat{R}_{i_1 j_1^* i_2 j_2^*}|_0$. Concrete expression of the coefficients in terms of the curvature tensor and its covariant derivatives up to the sixth order is given in [7] by

$$
\begin{aligned}
&\hat{K}(\omega,\omega^*)\\
&=\hat{K}|_0+\hat{F}(\omega)+\hat{F}^*(\omega^*)+\hat{g}_{ij^*}|_0\omega^i\omega^{*j}+\frac{1}{4}\hat{R}_{ij^*kl^*}|_0\omega^i\omega^k\omega^{*j}\omega^{*l}\\
&+\frac{1}{12}\hat{D}_m\hat{R}_{ij^*kl^*}|_0\omega^m\omega^i\omega^k\omega^{*j}\omega^{*l}+\frac{1}{12}\hat{D}_{m^*}\hat{R}_{ij^*kl^*}|_0\omega^i\omega^k\omega^{*j}\omega^{*l}\omega^{*m}\\
&+\frac{1}{24}\hat{D}_n\hat{D}_m\hat{R}_{ij^*kl^*}|_0\omega^n\omega^m\omega^i\omega^k\omega^{*j}\omega^{*l}+\frac{1}{24}\hat{D}_{n^*}\hat{D}_{m^*}\hat{R}_{ij^*kl^*}|_0\omega^i\omega^k\omega^{*j}\omega^{*l}\omega^{*m}\omega^{*n}\\
&+\frac{1}{36}\left(\hat{D}_{(n^*}\hat{D}_m\hat{R}_{ij^*kl^*)}+3\hat{g}^{or^*}\hat{R}_{o(j^*ml^*}\hat{R}_{in^*k)r^*}\right)|_0\omega^m\omega^i\omega^k\omega^{*j}\omega^{*l}\omega^{*n}+O(\omega^7)\,,
\end{aligned}
\tag{3.5}
$$

where $O(\omega^n)$ denotes terms of the order n in ω and ω^*. Here parentheses $(\cdots)$ enclosing indices denote the symmetrization of the holomorphic and anti-holomorphic indices, respectively, e.g. $T_{(i_1i_2\cdots i_Mj_1^*j_2^*\cdots j_N^*)}\equiv T_{(i_1i_2\cdots i_M)(j_1^*j_2^*\cdots j_N^*)}\equiv\frac{1}{N!M!}(T_{(i_1i_2\cdots i_Mj_1^*j_2^*\cdots j_N^*)}+T_{(i_2i_1\cdots i_Mj_1^*j_2^*\cdots j_N^*)}+T_{(i_1i_2\cdots i_Mj_2^*j_1^*\cdots j_N^*)}+T_{(i_2i_1\cdots i_Mj_2^*j_1^*\cdots j_N^*)}+\cdots)$.[2][3]

From the expansion of the Kähler potential (3.5) we can calculate the KNC expansion of the metric tensor through the fourth order, to yield

$$
\begin{aligned}
&\hat{g}_{ij^*}(\omega,\omega^*)\\
&=\hat{g}_{ij^*}|_0+\hat{R}_{ij^*kl^*}|_0\omega^k\omega^{*l}+\frac{1}{2}\hat{D}_m\hat{R}_{ij^*kl^*}|_0\omega^m\omega^k\omega^{*l}+\frac{1}{2}\hat{D}_{m^*}\hat{R}_{ij^*kl^*}|_0\omega^k\omega^{*l}\omega^{*m}\\
&+\frac{1}{6}\hat{D}_n\hat{D}_m\hat{R}_{ij^*kl^*}|_0\omega^n\omega^m\omega^k\omega^{*l}+\frac{1}{6}\hat{D}_{n^*}\hat{D}_{m^*}\hat{R}_{ij^*kl^*}|_0\omega^k\omega^{*l}\omega^{*m}\omega^{*n}\\
&+\frac{1}{4}\left(\hat{D}_{(n^*}\hat{D}_m\hat{R}_{ij^*kl^*)}+3\hat{g}^{or^*}\hat{R}_{o(j^*ml^*}\hat{R}_{in^*k)r^*}\right)|_0\omega^m\omega^k\omega^{*l}\omega^{*n}+O(\omega^5).
\end{aligned}
\tag{3.6}
$$

Compare this result with the metric expansion in RNC (2.11). Coefficients in both expansions are quite different. The relation between KNC and RNC expansions is discussed in Appendix B.

Let us discuss the inverse transformation of (3.3) in order to compare with the transformation laws (2.8) and (2.15) for RNC in Riemann and Kähler manifolds, respectively. We can show that the inverse transformation of the transformation

[2]It should be noted that we have changed notation from Ref. [7], in which we used the parentheses as a cyclic permutation without any numerical factor.

[3]All tensors in this expansion are symmetric in (anti-)holomorphic indices. We do not need the symmetrization symbol except for the last term, because of identities summarized in Appendix A. KNC are coordinates such that these identities become manifest.

(3.3) is given by

$$\pi^i = \omega^i - \sum_{N=2}^{\infty} \frac{1}{N!} \Gamma^i{}_{j_1 j_2 \cdots j_N}|_\varphi \omega^{j_1} \omega^{j_2} \cdots \omega^{j_N} \,. \tag{3.7}$$

Here $\Gamma^i{}_{j_1 j_2 \cdots j_N}$ is defined by

$$\Gamma^i{}_{j_1 j_2 \cdots j_N} = \nabla_{j_1} \nabla_{j_2} \cdots \nabla_{j_{N-2}} \Gamma^i{}_{j_{N-1} j_N} \tag{3.8}$$

in which ∇ is the covariant derivative acting on lower indices. Eq. (3.7) can be seen as follows: if we take π^i to be ω^i, we obtain the conditions $\Gamma^i{}_{(j_1 j_2 \cdots j_N)}|_\varphi = 0$ which are actually equivalent to the conditions (3.1) or (3.2) for KNC.

We summarize this subsection by remarking that

1. The transformation law (3.7) coincides with the restriction of (2.8) to holomorphic indices. In other words, only the differences between the transformation laws (2.15) for RNC and (3.7) for KNC in Kähler manifolds are non-holomorphic terms associated with covariant tensors. (See also the comments in subsection 2.2 and Eq. (B.2) in Appendix B.)

2. Using a holomorphic coordinate transformation, any choice of coordinates can be transformed to KNC, because the freedom in (3.7) and (3.3) coincide. However we cannot set $\Gamma^A{}_{(B_1 \cdots B_n)}|_\varphi = 0$ by any holomorphic coordinate transformations. This is why geodesics are not straight lines in KNC. (Compare this with footnote 1 in the case of RNC.)

Before closing this section, we give an example of KNC.

Ex.) A simple example of KNC is the standard coordinate in the Fubini-Study metric of $\mathbf{C}P^1$. Let z be a holomorphic coordinate. Then the Kähler potential can be written as

$$K(z, z^*) = \log(1 + |z|^2) \,. \tag{3.9}$$

By the equation $\partial_{z^*} \partial_z{}^N K = \frac{(-1)^{N+1} N! z^{*N-1}}{(1+|z|^2)^{N+1}}$, the condition (3.1) holds and therefore z is a KNC. Geodesics in KNC, $z(t) = \frac{\xi}{|\xi|} \tan(|\xi| t) = \xi t + \frac{1}{3}|\xi|^2 \xi t^3 + \cdots$, in which ξ is a tangent vector of the geodesic, are not linear in t.

3.2 The Transformation Law of Kähler Normal Coordinates

RNC in a Riemann manifold are defined by a tangent vector at the origin. However the geometric property of KNC has been obscure, since KNC are not defined by geodesics. The following theorem clarifies the geometric meaning of KNC.

Theorem

KNC transform like a *holomorphic tangent vector* at the origin of KNC,

$$\omega^i \to \omega'^i = \left.\frac{\partial z'^i}{\partial z^j}\right|_\varphi \omega^j , \tag{3.10}$$

under holomorphic coordinate transformations preserving $z^i = \varphi^i$, given by $\pi^i \to \pi'^i = \pi'^i(\pi) = c^i_{j_1}\pi^{j_1} + c^i_{j_1 j_2}\pi^{j_1}\pi^{j_2} + \cdots$.

A proof of this theorem is given in Appendix C.

This situation is quite different from RNC, because RNC transform as a tangent vector but they are not holomorphic in Kähler manifolds. From this theorem we find that there is one-to-one correspondence in the vicinity of the origin, between a point represented by KNC and a holomorphic tangent vector at the origin. Therefore the KNC are a quite natural extension of the RNC to the case of a Kähler manifold.

We can regard the expansion (3.5) as an expansion by a holomorphic tangent vector. Hence (3.5) is a *tensor equation* and holds for *any* holomorphic coordinates z^i because of the transformation law of (3.10). The expansion of the Kähler potential around $z^i = \varphi^i$ is given by

$$K(z,z^*) = K|_\varphi + F(\omega) + F^*(\omega^*) + g_{ij^*}|_\varphi \omega^i\omega^{*j} + \frac{1}{4}R_{ij^*kl^*}|_\varphi \omega^i\omega^k\omega^{*j}\omega^{*l} + \cdots . \tag{3.11}$$

Note that $z^i = \varphi^i$ represents the same point in the manifold with $\omega^i = 0$ in KNC.

3.3 The Kähler Normal Coordinate Expansion of Tensor Fields

In this subsection we discuss a covariant expansion of any tensor field using KNC. Any tensor $T_{i_1\cdots j_1^*\cdots}(z,z^*)$ can be expanded easily in KNC as

$$\hat{T}_{i_1\cdots j_1^*\cdots}(\omega,\omega^*) = \sum_{M,N=1}^{\infty} \frac{1}{M!N!}\hat{T}_{i_1\cdots j_1^*\cdots\, ,\, k_1\cdots k_M l_1^*\cdots l_N^*}|_0 \omega^{k_1}\cdots\omega^{k_M}\omega^{*l_1}\cdots\omega^{*l_N} , \tag{3.12}$$

where a hat denotes quantities in KNC. All coefficients are tensors in general holomorphic coordinates.

For an example, a vector with a holomorphic index $T_i(z, z^*)$ can be expanded as

$$\hat{T}_i(\omega,\omega^*) = \hat{T}_i|_0 + \hat{T}_{i,j}|_0\omega^j + \hat{T}_{i,k^*}|_0\omega^{*k} + \frac{1}{2}\hat{T}_{i,j_1j_2}|_0\omega^{j_1}\omega^{j_2} + \frac{1}{2}\hat{T}_{i,k_1^*k_2^*}|_0\omega^{*k_1}\omega^{*k_2} \\ +\hat{T}_{i,j_1k_1^*}|_0\omega^{j_1}\omega^{k_1^*} + O(\omega^3)\,. \tag{3.13}$$

Using (3.2), each coefficient can be rewritten as a covariant tensor in KNC, like

$$\hat{T}_{i,j}|_0 = \hat{D}_j\hat{T}_i|_0\,, \quad \hat{T}_{i,k^*}|_0 = \hat{D}_{k^*}\hat{T}_i|_0\,, \\ \hat{T}_{i,j_1j_2}|_0 = \hat{D}_{j_1}\hat{D}_{j_2}\hat{T}_i|_0\,, \quad \hat{T}_{i,k_1^*k_2^*}|_0 = \hat{D}_{k_1^*}\hat{D}_{k_2^*}\hat{T}_i|_0\,, \\ \hat{T}_{i,j_1k_1^*}|_0 = \hat{D}_{j_1}\hat{D}_{k_1^*}\hat{T}_i|_0\,(= \hat{D}_{k_1^*}\hat{D}_{j_1}\hat{T}_i|_0 + \hat{R}^l{}_{j_1k_1^*i}\hat{T}_l|_0)\,. \tag{3.14}$$

(Note that covariant expressions are not unique in general like in the last equation.) Hence the expansion of the tensor in terms of general holomorphic coordinates can be obtained as

$$T_i(z,z^*) = T_i|_\varphi + D_jT_i|_\varphi\omega^j + D_{k^*}T_i|_\varphi\omega^{*k} + \frac{1}{2}D_{j_1}D_{j_2}T_i|_\varphi\omega^{j_1}\omega^{j_2} \\ +\frac{1}{2}D_{k_1^*}D_{k_2^*}T_i|_\varphi\omega^{*k_1}\omega^{*k_2} + D_{j_1}D_{k_1^*}T_i|_\varphi\omega^{j_1}\omega^{k_1^*} + O(\omega^3)\,, \tag{3.15}$$

where we have not written any hats since this is a tensor equation from the theorem (3.10) and therefore is valid in any holomorphic coordinates (see subsection 3.2). In the case of a holomorphic vector $T_i(z)$ (for instance $T_i(z) = \partial_i W(z)$), this expansion reduces to

$$T_i(z) = T_i|_\varphi + D_jT_i|_\varphi\omega^j + \frac{1}{2}D_{j_1}D_{j_2}T_i|_\varphi\omega^{j_1}\omega^{j_2} + O(\omega^3)\,. \tag{3.16}$$

Actually expansion of any holomorphic tensor $T_{i_1\cdots i_M}(z)$ can be obtained to all orders:

$$T_{i_1\cdots i_M}(z) = \sum_{N=1}^{\infty}\frac{1}{N!}D_{j_1}\cdots D_{j_N}T_{i_1\cdots i_M}|_\varphi\omega^{j_1}\cdots\omega^{j_N}\,. \tag{3.17}$$

In the same way, a second rank tensor $T_{ij^*}(z,z^*)$ can be expanded like

$$T_{ij^*}(z,z^*) = T_{ij^*}|_\varphi + D_{k_1}T_{ij^*}|_\varphi\omega^{k_1} + D_{l_1^*}T_{ij^*}|_\varphi\omega^{*l_1} \\ +\frac{1}{2}D_{k_1}D_{k_2}T_{ij^*}|_\varphi\omega^{k_1}\omega^{k_2} + \frac{1}{2}D_{l_1^*}D_{l_1^*}T_{ij^*}|_\varphi\omega^{*l_1}\omega^{*l_2} \\ +(D_{l_1^*}D_{k_1}T_{ij^*} + R^m{}_{k_1l_1^*i}T_{mj^*})|_\varphi\omega^{k_1}\omega^{*l_1} + O(\omega^3)\,. \tag{3.18}$$

In the case of the metric tensor g_{ij^*}, this reduces to (3.6) in this order, because of the metric compatibility $D_k g_{ij^*} = D_{k^*} g_{ij^*} = 0$.

4 Kähler Normal Coordinates in the Background Field Method

In this section we apply the KNC expansion to the background field method in supersymmetric nonlinear sigma models. Target spaces of $D = 2, \mathcal{N} = 2$ (or $D = 4$, $\mathcal{N} = 1$) supersymmetric nonlinear sigma models must be Kähler manifolds [5]. The Lagrangian of supersymmetric nonlinear sigma models of scalar fields $A^i(x)$ and Weyl fermions $\psi^i(x)$ is given (after elimination of auxiliary fields) by (see [19])

$$\begin{aligned}\mathcal{L} = &-g_{ij^*}(A, A^*)\partial_\mu A^i \partial^\mu A^{*j} - i g_{ij^*}(A, A^*)\bar{\psi}^j \bar{\sigma}^\mu D_\mu \psi^i \\ &+\frac{1}{4} R_{ij^*kl^*}(A, A^*)\psi^i \psi^k \bar{\psi}^j \bar{\psi}^l \,,\end{aligned} \tag{4.1}$$

where the covariant derivative on the fermions is defined by $D_\mu \psi^i \equiv \partial_\mu \psi^i + \partial_\mu A^l \Gamma^i{}_{lk}(A, A^*)\psi^k$. Scalar fields are coordinates of a Kähler manifold. Under a holomorphic field redefinition of the scalar fields, $A^i \to A'^i = A'^i(A)$, the fermions and the quantity $\partial_\mu A^i(x)$ transform like holomorphic tangent vectors:

$$\psi^i(x) \quad \to \quad \psi'^i(x) = \frac{\partial A'^i}{\partial A^j}\psi^j(x) \,, \tag{4.2}$$

$$\partial_\mu A^i(x) \quad \to \quad \partial_\mu A'^i(x) = \frac{\partial A'^i}{\partial A^j}\partial_\mu A^j(x) \,. \tag{4.3}$$

By its definition, $D_\mu \psi^i$ also transforms as a holomorphic vector. Therefore the Lagrangian (4.1) is invariant under holomorphic coordinate transformations of the target space.

Next, we consider the background field method applied to supersymmetric nonlinear sigma models. A manifestly supersymmetric expansion of the Lagrangian by means of RNC (KNC) in terms of superfields is impossible. If we would promote transformation (2.8) or (3.3) to a relation between superfields, the connection in its transformation law depends on both holomorphic and anti-holomorphic coordinates of the background, and therefore chirality is not preserved [9].[4][5] We present the background field expansion in the Lagrangian (4.1) in component fields using KNC. To this end, we decompose the complex scalar fields $A^i(x)$ into the background fields $\varphi^i(x)$ and the fluctuating fields $\pi^i(x)$ around them:

$$A^i(x) = \varphi^i(x) + \pi^i(x)\,. \tag{4.4}$$

To expand the Lagrangian in terms of the fluctuations, we would like to transform $\pi^i(x)$ to KNC fields $\hat{\pi}^i(x)$. To do this, we must pay attention to the expansion of the kinetic term, because *the definition of the KNC depends on the space-time coordinates* through the background fields $\varphi^i(x)$ (see Eqs. (4.5) and (4.6), below). This was actually recognized in the RNC expansion in [1]. Here, we generalize the discussion in [1] to the case of KNC. Promoting (3.7) to a relation among fields, KNC fields $\hat{\pi}^i(x)$ can be expanded in terms of tangent vector fields $\hat{\omega}^i(x)$ as

$$\hat{\pi}^i(x) = \hat{\omega}^i(x) - \frac{1}{2}\hat{\Gamma}^i{}_{k_1k_2}|_\varphi \hat{\omega}^{k_1}(x)\hat{\omega}^{k_2}(x) + O(\hat{\omega}^3)\,, \tag{4.5}$$

where hats indicate quantities in KNC. When no space-time derivatives act on $\hat{\pi}^i$, the KNC fields $\hat{\pi}^i$ coincide with the tangent vector fields: $\hat{\pi}^i(x) = \hat{\omega}^i(x)$. Under the space-time derivative, however, the connection in (4.5) is also differentiated and remains non-zero:

$$\partial_\mu\hat{\pi}^i(x) = \hat{D}_\mu\hat{\omega}^i(x) - \frac{1}{2}\partial_\mu\varphi^{*j}(x)\hat{R}^i{}_{k_1j^*k_2}|_\varphi \hat{\omega}^{k_1}(x)\hat{\omega}^{k_2}(x) + O(\hat{\omega}^3)\,. \tag{4.6}$$

We have defined the covariant derivative on a tangent vector V^i at φ^i by $D_\mu V^i \equiv \partial_\mu V^i + \partial_\mu\varphi^j\Gamma^i{}_{jk}|_\varphi V^k$, and used the fact that it is simply $\hat{D}_\mu V^i = \partial_\mu V^i$ in KNC due to (3.2). For general holomorphic coordinates of fluctuations $\pi^i(x)$, Eq. (4.6) becomes

$$\partial_\mu\pi^i(x) = D_\mu\omega^i(x) - \frac{1}{2}\partial_\mu\varphi^{*j}(x)R^i{}_{k_1j^*k_2}|_\varphi \omega^{k_1}(x)\omega^{k_2}(x) + O(\omega^3)\,, \tag{4.7}$$

[4] At this time, we take the opportunity to correct an error in [7]. A manifestly supersymmetric expansion in KNC is impossible even around bosonic backgrounds. A superfield expansion in KNC is possible only in constant backgrounds. We would like to thank Thomas E. Clark for pointing this out.

[5] In the case of the Kähler manifold with isometry, an expansion in terms of superfields is given by Clark and Love in [10] defining new holomorphic quantities. We do not know its relation with KNC.

because of the transformation laws of (4.3) and (3.10). This is a tensor equation due to the theorem (3.10).

We have already given the KNC expansion of the metric in (3.6):

$$g_{ij^*}(\varphi+\pi,\varphi^*+\pi^*) = g_{ij^*}|_\varphi + R_{ij^*kl^*}|_\varphi \omega^k \omega^{*l} + O(\omega^3)\,. \tag{4.8}$$

Using Eqs. (4.7) and (4.8), we obtain the expansion of the bosonic kinetic term of the Lagrangian through second order in the fluctuations:

$$\begin{aligned}-\mathcal{L}_{\rm boson} &= g_{ij^*}|_\varphi \partial_\mu\varphi^i\partial^\mu\varphi^{*j} + g_{ij^*}|_\varphi (D_\mu\omega^i\partial^\mu\varphi^{*j} + \partial^\mu\varphi^i D_\mu\omega^{*j}) + g_{ij^*}|_\varphi D_\mu\omega^i D_\mu\omega^{*j}\\ &\quad + R_{ij^*kl^*}|_\varphi \left(\omega^k\omega^{*l}\partial_\mu\varphi^i\partial^\mu\varphi^{*j} - \frac{1}{2}\omega^i\omega^k\partial_\mu\varphi^{*j}\partial^\mu\varphi^{*l} - \frac{1}{2}\omega^{*j}\omega^{*l}\partial_\mu\varphi^i\partial^\mu\varphi^k\right)\\ &\quad + O(\omega^3)\,.\end{aligned} \tag{4.9}$$

Next we give the expansion of the fermion kinetic term. The expansion of the connection in KNC can be obtained from (3.5)

$$\begin{aligned}\Gamma^i{}_{lk}(\varphi+\pi,\varphi^*+\pi^*) &= R^i{}_{lk_1^*k}|_\varphi\omega^{*k_1} + \frac{1}{2}D_{k_2^*}R^i{}_{lk_1^*k}|_\varphi\omega^{*k_1}\omega^{*k_2}\\ &\quad + D_{j_1}R^i{}_{lk_1^*k}|_\varphi\omega^{j_1}\omega^{*k_1} + O(\omega^3)\,.\end{aligned} \tag{4.10}$$

Then the expansion of the fermion kinetic term through second order in ω can be determined to give

$$\begin{aligned}-\mathcal{L}_{\rm fermion} &= ig_{ij^*}|_\varphi\bar\psi^j\bar\sigma^\mu D_\mu\psi^i + iR_{lj^*kk_1^*}|_\varphi\partial_\mu\varphi^l(\bar\psi^j\bar\sigma^\mu\psi^k)\omega^{*k_1}\\ &\quad + iR_{ij^*j_1k_1^*}|_\varphi(\bar\psi^j\bar\sigma^\mu D_\mu\psi^i)\omega^{j_1}\omega^{*k_1} + iR_{j_1j^*kk_1^*}|_\varphi(\bar\psi^j\bar\sigma^\mu\psi^k)D_\mu\omega^{j_1}\omega^{*k_1}\\ &\quad + \frac{i}{2}D_{k_2^*}R_{lj^*kk_1^*}|_\varphi\partial_\mu\varphi^l(\bar\psi^j\bar\sigma^\mu\psi^k)\omega^{*k_1}\omega^{*k_2}\\ &\quad + iD_{j_1}R_{lj^*kk_1^*}|_\varphi\partial_\mu\varphi^l(\bar\psi^j\bar\sigma^\mu\psi^k)\omega^{j_1}\omega^{*k_2} + O(\omega^3)\,.\end{aligned} \tag{4.11}$$

Here we comment that

1. The expansions of (4.9) and (4.11) in KNC coincide with one in RNC in this order, since the difference in both coordinates appears in the third order as in Eq. (B.1). To preserve holomorphic structures beyond this order, we must use the KNC expansion given in this section.

2. In a constant background of $\partial_\mu\varphi^i = 0$, the expansion of (4.9) and (4.11) reduces to the expansion in [7].

Supersymmetric nonlinear sigma models admit a potential term [19]

$$\begin{aligned}\mathcal{L}_{\text{potential}} = &-g^{ij^*}(A,A^*)D_iW(A)D_{j^*}W(A^*)\\ &-\frac{1}{2}D_iD_jW(A)\psi^i\psi^j - \frac{1}{2}D_{i^*}D_{j^*}W^*(A^*)\bar{\psi}^i\bar{\psi}^j\,, \end{aligned} \tag{4.12}$$

where $W(A)$ is a holomorphic function called a superpotential. Using (3.17) and $g^{ij^*}(A,A^*) = g^{ij^*}|_\varphi + R^{ij^*}{}_{kl^*}|_\varphi\omega^k\omega^{*l} + O(\omega^3)$, the potential term can be expanded as

$$\begin{aligned}
&-\mathcal{L}_{\text{potential}} = [g^{ij^*}D_iWD_{j^*}W^*]_\varphi\\
&+[g^{ij^*}(D_{k_1}D_iW)D_{j^*}W^*]_\varphi\omega^{k_1} + [g^{ij^*}D_iW(D_{l_1^*}D_{j^*}W^*)]_\varphi\omega^{*l_1}\\
&+\frac{1}{2}[g^{ij^*}(D_{k_1}D_{k_2}D_iW)D_{j^*}W^*]_\varphi\omega^{k_1}\omega^{k_2} + \frac{1}{2}[g^{ij^*}D_iW(D_{l_1^*}D_{l_2^*}D_{j^*}W^*)]_\varphi\omega^{*l_1}\omega^{*l_2}\\
&+[g^{ij^*}(D_{k_1}D_iW)(D_{l_1^*}D_{j^*}W^*) + R^{ij^*}{}_{k_1l_1^*}D_iWD_{j^*}W^*]|_\varphi\omega^{k_1}\omega^{*l_1}\\
&+\frac{1}{2}(D_iD_jW|_\varphi + D_{k_1}D_iD_jW|_\varphi\omega^{k_1} + \frac{1}{2}D_{k_1}D_{k_2}D_iD_jW|_\varphi\omega^{k_1}\omega^{k_2})\psi^i\psi^j\\
&+\frac{1}{2}(D_{i^*}D_{j^*}W^*|_\varphi + D_{k_1^*}D_{i^*}D_{j^*}W^*|_\varphi\omega^{*l_1} + \frac{1}{2}D_{k_1^*}D_{k_2^*}D_{i^*}D_{j^*}W^*|_\varphi\omega^{*l_1}\omega^{*l_2})\bar{\psi}^i\bar{\psi}^j\\
&+O(\omega^3)\,.
\end{aligned} \tag{4.13}$$

5 Discussion

We discuss some applications of the KNC expansion in this section.

1. Nonlinear sigma models are renormalizable in two dimensions. Perturbative methods [6], however, cannot deal with a large coupling regime. On the contrary, the Wilsonian renormalization group (WRG) [11] can be applied in this region and may lead new results. To derive WRG equation, we need to expand the Lagrangian around the background field. Using the KNC expansion, we can derive the WRG equation generally covariant under the reparameterization of the background field. This would provides the further understanding of non-perturbative aspects of supersymmetric nonlinear sigma models, combined with non-perturbative analysis of Hermitian symmetric spaces using the large-N method [12] and related models on Ricci-flat Kähler manifolds [13].

2. In four dimensions, nonlinear sigma models can be considered as effective field theories of theories in higher energy scale such as supersymmetric QCD or

the minimal supersymmetric standard model. When symmetry is spontaneously broken down, there appear massless (quasi-)Nambu-Goldstone bosons in addition to fermionic superpartners [14]. Using the KNC expansion, low energy theorems of scattering amplitudes of these bosons have been discussed in [15]. A manifestly supersymmetric four derivative term with a rank four tensor has been recently found in Ref. [16]. Hence supersymmetric extension of the chiral perturbation theory would be possible, applying the KNC expansion of tensor fields in subsection 3.3 to higher rank tensors.

3. Some supersymmetric nonlinear sigma models with suitable potentials admit BPS domain wall solutions, which break the half of the original supersymmetry (see, e.g., [17]). The effective field theory on the wall is very interesting in the brane world scenario. To obtain this, we need to expand the Lagrangian around the domain wall background, as was done in the cases of the linear models in [18]. We will find that the KNC expansion [with a potential term (4.13)] is a very powerful tool to construct effective field theories on BPS domain walls in supersymmetric nonlinear sigma models.

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A Kähler Manifolds

In this appendix, we summarize geometry of Kähler manifolds. The uppercase Roman letters are used for both holomorphic and anti-holomorphic indices: $\{x^A\} =$

$\{z^i, z^{*i}\}$. The the complex structure J and the Hermitian metric g are given on Hermitian manifolds. The Kähler form $\Omega \equiv i g_{ij^*} dz^i \wedge dz^{*j}$ is closed on Kähler manifolds: $d\Omega = 0$. From this condition, the metric can be written as

$$g_{ij^*}(z, z^*) = \frac{\partial^2 K(z, z^*)}{\partial z^i \partial z^{*j}} = K_{,ij^*}(z, z^*) \tag{A.1}$$

using a real function K called the Kähler potential. There exists freedom of redefinition of K called a Kähler transformation: $K' = K + f(z) + f^*(z^*)$ with arbitrary holomorphic function f. Components of the affine connection with mixed indices disappear as a result of the compatibility condition of the complex structure, $DJ = 0$. Non-zero components of the connection are

$$\Gamma^k{}_{ij} = g^{kl^*} g_{jl^*,i} = g^{kl^*} K,_{ijl^*} \tag{A.2}$$

and their conjugates. Independent components of the curvature tensor are

$$R^{i^*}{}_{j^*kl^*} = \partial_k \Gamma^{i^*}{}_{j^*l^*} = \partial_k (g^{mi^*} g_{mj^*,l^*}) \tag{A.3}$$

and its conjugate. The curvature tensor with lower indices

$$R_{ij^*kl^*} \equiv g_{im^*} R^{m^*}{}_{j^*kl^*} = K,_{ij^*kl^*} - g^{mn^*} K,_{mj^*l^*} K,_{n^*ik} \tag{A.4}$$

has some symmetries among indices. In addition to the symmetries of the curvature tensor on Riemann manifolds,

$$R_{ABCD} = -R_{ABDC} = -R_{BACD} = R_{CDAB}\,, \tag{A.5}$$

there exist further symmetries

$$R_{ij^*kl^*} = R_{kj^*il^*} = R_{il^*kj^*} \tag{A.6}$$

as a result of the Kähler condition. The Bianchi identity $D_A R_{BCDE} + D_C R_{ABDE} + D_B R_{CADE} = 0$ on Riemann manifolds reduces to

$$D_m R_{ij^*kl^*} = D_i R_{mj^*kl^*} \tag{A.7}$$

in the case of Kähler manifolds. Commutators of covariant derivatives on arbitrary tensor $T_{C_1 \cdots C_n}$ is given by

$$[D_A, D_B]\, T_{C_1 \cdots C_n} = \sum_{a=1}^{n} R_{ABC_a}{}^{D} T_{C_1 \cdots C_{a-1} D C_{a+1} \cdots C_n}\,. \tag{A.8}$$

Equations $[D_i, D_j] = [D_{i^*}, D_{j^*}] = 0$ hold as a result of the Kähler property. KNC are coordinates such that these identities become manifest.

B Relation Between Kähler and Riemann Normal Coordinates

In section 2.2, we have considered geodesics in general holomorphic coordinates. Considering geodesics in KNC ω^i instead of general coordinates $\pi^i = z^i - \varphi^i$, we can obtain the relation of KNC and RNC. Their relation up to the fourth order is obtained instead of (2.15), to give

$$\omega^i = \xi^i - \frac{1}{3!}\hat{R}^i{}_{j_1 k_1^* j_2}|_0 \xi^{j_1}\xi^{j_2}\xi^{*k_1} - \frac{2}{4!}\hat{D}_{j_1}\hat{R}^i{}_{j_2 k_1^* j_3}|_0 \xi^{j_1}\xi^{j_2}\xi^{j_3}\xi^{*k_1}$$
$$-\frac{1}{4!}\hat{D}_{k_1^*}\hat{R}^i{}_{j_1 k_2^* j_2}|_0 \xi^{j_1}\xi^{j_2}\xi^{*k_1}\xi^{*k_2} + O(\xi^5)\,, \tag{B.1}$$

in which a hat denotes quantities in KNC, and the condition (3.2) has been used. In general, all coefficients of the expansion of the transformation from RNC to KNC are covariant tensors T composed of the curvature and metric tensors and their covariant derivatives because of (3.2):

$$\omega^i = \xi^i - \sum_{M=2,N=1}^{\infty} \hat{T}^i{}_{j_1\cdots j_M k_1^*\cdots k_N^*}(\hat{D},\hat{R},\hat{g})|_0 \xi^{j_1}\cdots\xi^{j_M}\xi^{*k_1}\cdots\xi^{*k_N}\,. \tag{B.2}$$

Here let us make some comments.

1. KNC and RNC coincide if and only if the Kähler manifold is flat.

2. In the cases of Hermitian symmetric spaces, the equation $DR = 0$ holds. Therefore the tensors T in (B.2) are composed of only the curvature and metric tensors.

Next we show the relation between KNC and RNC in the expansion of the metric tensor in both coordinates. From the transformation law (B.1), the Jacobian can be calculated, to give

$$\frac{\partial\omega^i}{\partial\xi^l} = \delta^i_l - \frac{1}{3}\hat{R}^i{}_{j_1 k_1^* l}|_0 \xi^{j_1}\xi^{*k_1} - \frac{1}{4}\hat{D}_{j_1}\hat{R}^i{}_{j_2 k_1^* l}|_0 \xi^{j_1}\xi^{j_2}\xi^{*k_1}$$
$$-\frac{1}{12}\hat{D}_{k_1^*}\hat{R}^i{}_{j_1 k_2^* l}|_0 \xi^{j_1}\xi^{*k_1}\xi^{*k_2} + O(\xi^4)\,,$$
$$\frac{\partial\omega^i}{\partial\xi^{*l}} = -\frac{1}{6}\hat{R}^i{}_{j_1 l^* j_2}|_0 \xi^{j_1}\xi^{j_2} - \frac{1}{12}\hat{D}_{j_1}\hat{R}^i{}_{j_2 l^* j_3}|_0 \xi^{j_1}\xi^{j_2}\xi^{j_3}$$
$$-\frac{1}{12}\hat{D}_{k_1^*}\hat{R}^i{}_{j_1 l^* j_2}|_0 \xi^{j_1}\xi^{j_2}\xi^{*k_1} + O(\xi^4)\,. \tag{B.3}$$

The KNC expansion of the metric (3.6) is transformed to

$$
\begin{aligned}
\bar{g}_{ij^*} &= \hat{g}_{ij^*}|_0 - \frac{1}{3}\hat{R}_{ij^*kl^*}|_0\xi^k\xi^{*l} - \frac{1}{6}\hat{D}_m\hat{R}_{ij^*kl^*}|_0\xi^m\xi^k\xi^{*l} \\
&\quad -\frac{1}{6}\hat{D}_{m^*}\hat{R}_{ij^*kl^*}|_0\xi^k\xi^{*l}\xi^{*m} + O(\xi^4) \\
\bar{g}_{ij} &= -\frac{1}{3}\hat{R}_{ik^*jl^*}|_0\xi^{*k}\xi^{*l} - \frac{1}{6}\hat{D}_{k^*}\hat{R}_{il^*jm^*}|_0\xi^{*k}\xi^{*l}\xi^{*m} \\
&\quad -\frac{1}{6}\hat{D}_k\hat{R}_{il^*jm^*}|_0\xi^k\xi^{*l}\xi^{*m} + O(\xi^4)\,,
\end{aligned}
\tag{B.4}
$$

where a bar denotes the tensors in RNC. Note that the tensors in the right hand sides in these equations are ones in KNC (or general holomorphic coordinates). There appear non-Hermitian components, $\bar{g}_{ij}$ and its conjugate, since the transformation (B.2) is not holomorphic. These components (B.4) coincide with the RNC expansion of the metric (2.11), identifying real coordinates as $\{x^A\} = \{z^i, z^{*i}\}$ and enforcing the Kähler condition on the curvature tensor, like $R_{ijkl} = R_{ij^*kl} = R_{ijkl^*} = 0$, in the right hand side of (2.11). We would like to emphasize again that the RNC expansion of the metric includes unwanted non-Hermitian terms.

C A Proof of the Theorem

In this appendix, we would like to show that KNC in a Kähler manifold can be interpreted as a holomorphic tangent vector at the origin, therefore they are natural extension of the RNC to the case of a Kähler manifold. To this end, we would like to discuss the relation between a set of KNC defined by a set of general holomorphic coordinates z^i and $z'^i = z'^i(z)$, which are transformed under a holomorphic coordinate transformation preserving the origin. (In this appendix, we take the expansion point to be the origin, $\varphi^i = 0$, for simplicity, but all discussions hold for general expansion points, replacing z^i to $\pi^i = z^i - \varphi^i$.)

First, we need the transformation law of the "generalized connection" $K_{,j^*i_1\cdots i_N}(z,z^*)$ in the definition of KNC, denoted by the following lemma.

Lemma

The transformation law of $K_{,j^*i_1\cdots i_N}(z,z^*)$ under a holomorphic coordinate transformation, $z^i \to z'^i = z'^i(z)$, is given by

$$
\begin{aligned}
& K_{,j^* i_1 \cdots i_N}(z, z^*) \\
\to \quad & K_{,j'^* i'_1 \cdots i'_N}(z', z'^*) = \sum_{n=1}^{N} \frac{1}{n!} K_{,l^* k_1 \cdots k_n}(z, z^*) \frac{\partial z^{*l}}{\partial z'^{*j}} \left[\frac{\partial^N (z^{k_1} \cdots z^{k_n})}{\partial z'^{i_1} \cdots \partial z'^{i_N}} \right]_* , \quad \text{(C.1)}
\end{aligned}
$$

where $[\cdots]_*$ represents that terms including z differentiated by no z' should be omitted.

The term of $n = N$ is a homogeneous (tensorial) term, but all of the other terms are non-homogeneous terms. The $N = 2$ case corresponds to the ordinary connection: $K_{,j^* i_1 i_2} = g_{kj^*} \Gamma^k_{i_1 i_2}$.

(Proof) We use the mathematical induction for a proof.

i) The $N = 1$ case. Eq. (C.1) in the case of $N = 1$ is

$$
K_{,j^* i_1} \to K_{,j'^* i'_1} = K_{,l^* k_1} \frac{\partial z^{*l}}{\partial z'^{*j}} \frac{\partial z^{k_1}}{\partial z'^{i_1}} . \quad \text{(C.2)}
$$

This is obvious because of $K_{,j^* i_1} = g_{i_1 j^*}$.

ii) We assume that Eq. (C.1) holds for N. Differentiation of Eq. (C.1) with respect to $z'^{i_{N+1}}$ can be calculated, to yield

$$
\begin{aligned}
K_{,j'^* i'_1 \cdots i'_{N+1}} = & \sum_{n=1}^{N} \frac{1}{n!} K_{,l^* k_1 \cdots k_{n+1}} \frac{\partial z^{*l}}{\partial z'^{*j}} \frac{\partial z^{k_{n+1}}}{\partial z'^{i_{N+1}}} \left[\frac{\partial^N (z^{k_1} \cdots z^{k_n})}{\partial z'^{i_1} \cdots \partial z'^{i_N}} \right]_* \\
& + \sum_{n=1}^{N} \frac{1}{n!} K_{,l^* k_1 \cdots k_n} \frac{\partial z^{*l}}{\partial z'^{*j}} \frac{\partial}{\partial z'^{i_{N+1}}} \left[\frac{\partial^N (z^{k_1} \cdots z^{k_n})}{\partial z'^{i_1} \cdots \partial z'^{i_N}} \right]_* . \quad \text{(C.3)}
\end{aligned}
$$

The first term can be rewritten as

$$
\sum_{n=2}^{N+1} \frac{1}{(n-1)!} K_{,l^* k_1 \cdots k_n} \frac{\partial z^{*l}}{\partial z'^{*j}} \frac{\partial z^{k_n}}{\partial z'^{i_{N+1}}} \left[\frac{\partial^N (z^{k_1} \cdots z^{k_{n-1}})}{\partial z'^{i_1} \cdots \partial z'^{i_N}} \right]_* . \quad \text{(C.4)}
$$

Therefore we have

$$
\begin{aligned}
K_{,j'^* i'_1 \cdots i'_{N+1}} = & K_{,l^* k_1} \frac{\partial z^{*l}}{\partial z'^{*j}} \frac{\partial^{N+1} z^{k_1}}{\partial z'^{i_1} \cdots \partial z'^{i_{N+1}}} \\
& + \sum_{n=2}^{N} \frac{1}{n!} K_{,l^* k_1 \cdots k_n} \frac{\partial z^{*l}}{\partial z'^{*j}} \left\{ n \frac{\partial z^{k_n}}{\partial z'^{i_{N+1}}} \left[\frac{\partial^N (z^{k_1} \cdots z^{k_{n-1}})}{\partial z'^{i_1} \cdots \partial z'^{i_N}} \right]_* \right. \\
& \qquad \left. + \frac{\partial}{\partial z'^{i_{N+1}}} \left[\frac{\partial^N (z^{k_1} \cdots z^{k_n})}{\partial z'^{i_1} \cdots \partial z'^{i_N}} \right]_* \right\} \\
& + \frac{1}{N!} K_{,l^* k_1 \cdots k_{N+1}} \frac{\partial z^{*l}}{\partial z'^{*j}} \frac{\partial z^{k_{N+1}}}{\partial z'^{i_{N+1}}} \left[\frac{\partial^N (z^{k_1} \cdots z^{k_N})}{\partial z'^{i_1} \cdots \partial z'^{i_N}} \right]_* . \quad \text{(C.5)}
\end{aligned}
$$

In the brace of the right hand side, the relation

$$
\begin{aligned}
&n\frac{\partial z^{k_n}}{\partial z'^{i_{N+1}}}\left[\frac{\partial^N(z^{k_1}\cdots z^{k_{n-1}})}{\partial z'^{i_1}\cdots\partial z'^{i_N}}\right]_* + \frac{\partial}{\partial z'^{i_{N+1}}}\left[\frac{\partial^N(z^{k_1}\cdots z^{k_n})}{\partial z'^{i_1}\cdots\partial z'^{i_N}}\right]_* \\
&= \left[\frac{\partial^{N+1}(z^{k_1}\cdots z^{k_n})}{\partial z'^{i_1}\cdots\partial z'^{i_{N+1}}}\right]_* \qquad \text{(C.6)}
\end{aligned}
$$

holds, where the symmetrization of the first term in the left hand side is implied. We thus obtain

$$
K_{,j'^*i'_1\cdots i'_{N+1}} = \sum_{n=1}^{N+1}\frac{1}{n!}K_{,l^*k_1\cdots k_n}\frac{\partial z^{*l}}{\partial z'^{*j}}\left[\frac{\partial^{N+1}(z^{k_1}\cdots z^{k_n})}{\partial z'^{i_1}\cdots\partial z'^{i_{N+1}}}\right]_* . \qquad \text{(C.7)}
$$

iii) From i) and ii) we have proved the lemma. (Q.E.D.)

We call a holomorphic coordinate transformation, which leaves the origin invariant (i.e. $z^i = 0$ implies $z'^i = 0$ and vice versa), a holomorphic coordinate transformation preserving the origin. We immediately obtain the following corollary from the lemma:

Corollary

Under holomorphic coordinate transformations preserving the origin, $K_{,j^*i_1\cdots i_N}$ transforms like

$$
\begin{aligned}
K_{,j^*i_1\cdots i_N}|_0 \to K_{,j'^*i'_1\cdots i'_N}|_0 &= \sum_{n=1}^{N}\frac{1}{n!}K_{,l^*k_1\cdots k_n}|_0\left[\frac{\partial z^{*l}}{\partial z'^{*j}}\frac{\partial^N(z^{k_1}\cdots z^{k_n})}{\partial z'^{i_1}\cdots\partial z'^{i_N}}\right]_0 \\
&= \sum_{n=1}^{\infty}\frac{1}{n!}K_{,l^*k_1\cdots k_n}|_0\left[\frac{\partial z^{*l}}{\partial z'^{*j}}\frac{\partial^N(z^{k_1}\cdots z^{k_n})}{\partial z'^{i_1}\cdots\partial z'^{i_N}}\right]_0 \qquad \text{(C.8)}
\end{aligned}
$$

at the origin, where the subscripts "0" denote the values evaluated at the origin of the coordinates, $z^i = 0$ or $z'^i = 0$. The second equality holds because the term $[\cdots]_0$ vanishes when $n > N$.

We are now ready to prove the theorem, which reveals the geometric meaning of KNC.

(A proof of the theorem) Using the definition (3.3) and the corollary (C.8), the left hand side of Eq. (3.10) can be explicitly calculated, to give

$$
\begin{aligned}
\omega'^i &= \sum_{n=1}^{\infty}\frac{1}{n!}(g'^{ij^*}K_{,j'^*i'_1\cdots i'_n})_0 z'^{i_1}\cdots z'^{i_n} \\
&= \sum_{n=1}^{\infty}\frac{1}{n!}g'^{ij^*}|_0\left(\sum_{m=1}^{\infty}\frac{1}{m!}K_{,l^*k_1\cdots k_m}|_0\left[\frac{\partial z^{*l}}{\partial z'^{*j}}\frac{\partial^n(z^{k_1}\cdots z^{k_m})}{\partial z'^{i_1}\cdots\partial z'^{i_n}}\right]_0\right) z'^{i_1}\cdots z'^{i_n}
\end{aligned}
$$

$$= \left.\frac{\partial z'^i}{\partial z^k}\right|_0 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n!m!}(g^{kl^*}K_{,l^*k_1\cdots k_m})_0\left[\frac{\partial^n(z^{k_1}\cdots z^{k_m})}{\partial z'^{i_1}\cdots\partial z'^{i_n}}\right]_0 z'^{i_1}\cdots z'^{i_n}$$

$$= \left.\frac{\partial z'^i}{\partial z^k}\right|_0 \sum_{m=1}^{\infty}\frac{1}{m!}(g^{kj^*}K_{,j^*k_1\cdots k_m})_0 z^{k_1}\cdots z^{k_m} = \left.\frac{\partial z'^i}{\partial z^k}\right|_0 \omega^k\,. \tag{C.9}$$

(Q.E.D.)

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